CELL MEANS MODELS FOR THE 2-WAY CLASSIFICATION MIXED MODEL

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S. R. Searle

Biometrics Unit, Cornell University, Ithaca, New York

Abstract

The cell means model is applied to the 2-way crossed classification mixed model using generalized least squares. The general case of unequal-subclass-numbers data is considered, including the possibility of having some empty cells; and application to split plots and to balanced incomplete blocks is shown.

1. Introduction

The cell means model has for several years received notable attention in the literature (e.g., Speed, Hocking and Hackney, 1978, and Urquhart and Weeks, 1978) as a useful way of handling linear models. This is particularly so in situations of unequal-subclass-numbers data (unbalanced data) and where interactions are to be part of the model, especially if some cells of the data are empty — i.e., contain no data. Recently, however, Steinhorst (1982) has cast doubt on the adaptability of the cell means model to mixed models. In connection with a randomized complete blocks

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situation he writes that he is "··· at a loss to see how μ_{ij} carries the right meaning if blocks are random ···". And regarding the "split-plot design or a random or mixed model" he continues "The cell-means model is not of much help in such cases. The classic split-plot model ··· cannot be replaced by a variation of $y_{ijk} = \mu_{ijk} + e_{ijk}$." It is the purpose of this paper to show that this negative attitude to the cell means model is not correct. All of the cases (and more) referred to can be shown to fit perfectly into the cell-means-model framework. Furthermore, for the randomized complete blocks model with random blocks (as is usual), extension to unbalanced data is quite feasible. An explicit (matrix-vector) expression is developed for estimating the treatment means.

a. Data description

We consider the 2-way cross-classification in terms of a rows-by-columns layout, having a rows and b columns. The number of observations in the cell defined by row i and column j is denoted by n_{ij} for $i = 1,\cdots,a$ and $j = 1,\cdots,b$. Balanced data means data in which $n_{ij} = n$ for all i and j, the simplest case of which is $n = 1$. Unbalanced data means data in which the n_{ij} are not all the same and indeed some of them may be zero, i.e., $n_{ij} \geq 0$. When the cell defined by row i and column j contains data its k'th observation is denoted by y_{ijk} for $k = 1,2,\cdots,n_{ij}$. Any cell having no data is said to be empty, and $n_{ij} = 0$.

We use the customary notation for totals and means, viz.

$$y_{ij.} = \sum_{k=1}^{n_{ij}} y_{ijk} \quad \text{with} \quad \bar{y}_{ij.} = y_{ij.}/n_{ij} ,$$

$$y_{i..} = \sum_{j=1}^{b} y_{ij.} \quad \text{with} \quad \bar{y}_{i..} = y_{i..}/n_{i.} ,$$

and

$$y_{...} = \sum_{i=1}^{a} y_{i..} \quad \text{with} \quad \bar{y}_{...} = y_{...}/n_{..} ,$$

where $n_{i\cdot} = \sum_{j=1}^{b} n_{ij}$ and $n_{\cdot\cdot} = \sum_{i=1}^{a} n_{i\cdot}$; and $y_{\cdot j\cdot}$, $\bar{y}_{\cdot j\cdot}$ and $n_{\cdot j}$ are defined similarly. The observations are deemed to be arrayed in a column vector $\mathbf{y}$ in lexicon order; it is represented as

$$\mathbf{y} = \left\{\left\{\left\{y_{ijk}\right\}_{k=1}^{k=n_{ij}}\right\}_{j=1}^{j=b}\right\}_{i=1}^{i=a}$$

showing that subscript k changes fastest, then j and then i.

Example 1: For data in the form of Grid 1

Grid 1

$n_{11} = 1$	$n_{12} = 2$	$n_{22} = 3$
$n_{21} = 3$	$n_{22} = 4$	$n_{23} = 1$

the row vector form of $\mathbf{y}$ is

$$\mathbf{y}' = [y_{111}\ y_{121}\ y_{122}\ y_{131}\ y_{132}\ y_{133}\ y_{211}\ y_{212}\ y_{213}\ y_{221}\ y_{222}\ y_{223}\ y_{224}\ y_{231}].$$

b. Summing vectors and J-matrices

Considerable use is made of summing vectors, e.g., $\mathbf{1}_r$ is a vector of r unities; and of corresponding matrices, $\mathbf{J}_r = \mathbf{1}_r\mathbf{1}_r'$, a square matrix of order r having every element unity; and $\mathbf{J}_{r,s} = \mathbf{1}_r\mathbf{1}_s'$, a matrix of order $r \times s$ with every element unity. The result

$$(a\mathbf{I}_n + b\mathbf{J}_n)^{-1} = \frac{1}{a}(\mathbf{I}_n - \frac{b}{a + nb}\mathbf{J}_n) \quad \text{for } a \neq 0 \text{ and } a \neq bn$$

is also used.

c. Direct products and sums of matrices

The direct product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is defined as $\mathbf{A} \otimes \mathbf{B} = \{a_{ij}\mathbf{B}\}$. It is a useful operation in many cases of balanced data (e.g., Searle and Henderson, 1979), two of which are dealt with here. Properties of the operator are

$$\begin{aligned} (\underset{\sim}{A} \otimes \underset{\sim}{B})' &= \underset{\sim}{A}' \otimes \underset{\sim}{B}' & (\underset{\sim}{A} \otimes \underset{\sim}{B})(\underset{\sim}{X} \otimes \underset{\sim}{Y}) &= \underset{\sim}{AX} \otimes \underset{\sim}{BY} \\ (\underset{\sim}{A} \otimes \underset{\sim}{B})^{-1} &= \underset{\sim}{A}^{-1} \otimes \underset{\sim}{B}^{-1} & \underset{\sim}{A}^{-1}(\underset{\sim}{A} \otimes \underset{\sim}{1}') &= (\underset{\sim}{A}^{-1} \otimes 1)(\underset{\sim}{A} \otimes \underset{\sim}{1}') = \underset{\sim}{I} \otimes \underset{\sim}{1}' \end{aligned} \tag{1}$$

where conformability for the results is assumed to hold. We also use the direct sum:

$$\underset{\sim}{A} \oplus \underset{\sim}{B} = \begin{Bmatrix} \underset{\sim}{A} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B} \end{Bmatrix}$$

especially in its more general form

$$\bigoplus_{i=1}^{t} \underset{\sim}{A}_i = \underset{\sim}{A}_1 + \underset{\sim}{A}_2 + \cdots + \underset{\sim}{A}_t = \begin{bmatrix} \underset{\sim}{A}_1 & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{A}_2 & & \vdots \\ \vdots & & \ddots & \\ \underset{\sim}{0} & \cdots & & \underset{\sim}{A}_t \end{bmatrix} .$$

This is a diagonal matrix when the A_i's are scalars; e.g.,

$$\bigoplus_{i=1}^{3} a_i = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} .$$

d. Generalized least squares

An important distinction between fixed and mixed models is that for the former the variance-covariance matrix of the vector of observations is $\sigma_e^2 \underset{\sim}{I}$ whereas for mixed models it has a form different from $\sigma_e^2 \underset{\sim}{I}$. This arises from covariances that are deemed to be part of the model; e.g., in the randomized complete blocks situation a covariance between observations in the same block. As a result, we denote the variance-covariance matrix of $\underset{\sim}{y}$ quite generally by $\underset{\sim}{V}$ and deal with a linear model

$$E(\underset{\sim}{y}) = \underset{\sim}{X}\underset{\sim}{\mu} \quad \text{and} \quad \text{var}(\underset{\sim}{y}) = \underset{\sim}{V} \tag{2}$$

where $E(\underset{\sim}{y})$ is the expected value of $\underset{\sim}{y}$ over repeated sampling, $\underset{\sim}{\mu}$ is the vector of parameters to be estimated (in our case cell means that have to be specified in each case) and $\underset{\sim}{X}$ is the known incidence matrix of zeros and unities corresponding to the occurrence of the elements of $\underset{\sim}{\mu}$ in $E(\underset{\sim}{y})$. Models where columns of $\underset{\sim}{X}$ are covariables can also be accommodated but shall not be considered here.

$\underset{\sim}{V}$ is assumed to be non-singular. (Singular $\underset{\sim}{V}$ can be accommodated but it, too, is not dealt with here.) Then the well-known generalized least squares (GLS) equations for estimating $\underset{\sim}{\mu}$, sometimes also called the Aitken equations, are

$$\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X}\hat{\underset{\sim}{\mu}} = \underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{y} \ . \tag{3}$$

2. Randomized Complete Blocks

Procedures for applying the cell means model to the 2-way classification mixed model are introduced by considering the easy case of randomized complete blocks. Thinking of columns as being the blocks, with one observation on every treatment (row) in each block, we have $k = 1$ for every cell and represent y_{ij1} as y_{ij}. Then

$$\underset{\sim}{y}' = [y_{11} \cdots y_{1b} \cdots y_{i1} \cdots y_{ij} \cdots y_{ib} \cdots y_{a1} \cdots y_{ab}]$$

and representation in terms of a cell means model is

$$E(y_{ij}) = \mu_i \ ;$$

i.e., μ_i for $i = 1,\cdots,a$ are the row means that are to be estimated.

Example 2: For a = 2 and b = 3

$$\underset{\sim}{y} = \begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{23} \end{bmatrix} \quad \text{and} \quad E(\underset{\sim}{y}) = \begin{bmatrix} 1 & \cdot \\ 1 & \cdot \\ 1 & \cdot \\ \cdot & 1 \\ \cdot & 1 \\ \cdot & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} .$$

In general, for $\underset{\sim}{\mu} = [\mu_1 \; \mu_2 \; \cdots \; \mu_i \; \cdots \; \mu_a]'$

$$E(\underset{\sim}{y}) = (\underset{\sim}{I}_a \otimes \underset{\sim}{1}_b)\underset{\sim}{\mu}$$

so that $\underset{\sim}{X}$ of (2) is

$$\underset{\sim}{X} = (\underset{\sim}{I}_a \otimes \underset{\sim}{1}_b) \; . \tag{4}$$

To use the GLS equations of (3) we need $\underset{\sim}{V}$. This requires defining the variance and covariance structure of the elements of $\underset{\sim}{y}$. Since in randomized blocks analysis block effects are taken to be random variables, we specify that in addition to σ_e^2 being common to the variance of each observation, any pair of observations in the same block has a covariance σ_β^2. Thus the variance of y_{ij} is

$$v(y_{ij}) = \sigma_e^2 + \sigma_\beta^2 \tag{5}$$

and

$$\text{cov}(y_{ij}, y_{i'j}) = \sigma_\beta^2 \quad \text{for } i' \neq i \; .$$

and (6)

$$\text{cov}(y_{ij}, y_{i'j'}) = 0 \quad \text{for } j \neq j' \; .$$

Notation: For simplicity write

$$e \equiv \sigma_e^2 \quad \text{and} \quad \beta \equiv \sigma_\beta^2 .$$

Example 2 (continued):

$$\text{var}(\underset{\sim}{y}) = \begin{bmatrix} e+\beta & \cdot & \cdot & \beta & \cdot & \cdot \\ \cdot & e+\beta & \cdot & \cdot & \beta & \cdot \\ \cdot & \cdot & e+\beta & \cdot & \cdot & \beta \\ \beta & \cdot & \cdot & e+\beta & \cdot & \cdot \\ \cdot & \beta & \cdot & \cdot & e+\beta & \cdot \\ \cdot & \cdot & \beta & \cdot & \cdot & e+\beta \end{bmatrix} = e\underset{\sim}{I} + \beta \begin{bmatrix} 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 \end{bmatrix}$$

$$= e\underset{\sim}{I}_6 + \beta(\underset{\sim}{J}_2 \otimes \underset{\sim}{I}_3) .$$

Generalization to a treatments and b blocks yields

$$\underset{\sim}{V} = e\underset{\sim}{I}_{ab} + \beta(\underset{\sim}{J}_a \otimes \underset{\sim}{I}_b) = (e\underset{\sim}{I}_a + \beta\underset{\sim}{J}_a) \otimes \underset{\sim}{I}_b . \tag{7}$$

The first expression needed for the GLS equations is $\underset{\sim}{V}^{-1}$ and, applying (1) to (7) this is

$$\underset{\sim}{V}^{-1} = (e\underset{\sim}{I}_a + \beta\underset{\sim}{J}_a)^{-1} \otimes \underset{\sim}{I}_b = \frac{1}{e}\left(\underset{\sim}{I}_a - \frac{\beta}{e + a\beta}\underset{\sim}{J}_a\right) \otimes \underset{\sim}{I}_b . \tag{8}$$

Hence, with $\underset{\sim}{X}$ of (4) and using (1) again

$$\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X} = (\underset{\sim}{I}_a \otimes \underset{\sim}{1}_b')\left[\frac{1}{e}\left(\underset{\sim}{I}_a - \frac{\beta}{e + a\beta}\underset{\sim}{J}_a\right) \otimes \underset{\sim}{I}_b\right](\underset{\sim}{I}_a \otimes \underset{\sim}{1}_b)$$

$$= (b/e)\left(\underset{\sim}{I}_a - \frac{\beta}{e + a\beta}\underset{\sim}{J}_a\right) .$$

Also,

$$\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{y} = (\underset{\sim}{I}_a \otimes \underset{\sim}{1}_b')\left[\frac{1}{e}\left(\underset{\sim}{I}_a - \frac{\beta}{e + a\beta}\underset{\sim}{J}_a\right) \otimes \underset{\sim}{I}_b\right]\underset{\sim}{y}$$

$$= \frac{1}{e}\left(\underset{\sim}{I}_a - \frac{\beta}{e + a\beta}\underset{\sim}{J}_a\right) \otimes \underset{\sim}{1}_b'\Big]\underset{\sim}{y} .$$

Hence from (3)

$$\begin{aligned}\hat{\underset{\sim}{\mu}} &= (\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{y} \\ &= \frac{1}{b}\left[\frac{1}{e}\left(\underset{\sim}{I}_a - \frac{\beta}{e + a\beta}\right)\right]^{-1}\left[\frac{1}{e}\left(\underset{\sim}{I}_a - \frac{\beta}{e + a\beta}\right)\otimes \underset{\sim}{1}_b'\right]\underset{\sim}{y} \\ &= \frac{1}{b}(\underset{\sim}{I}_a \otimes \underset{\sim}{1}_b')\underset{\sim}{y} \ .\end{aligned} \tag{9}$$

Then, on observing for the example that (9) is

$$\begin{bmatrix}\hat{\mu}_1 \\ \hat{\mu}_2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{23}\end{bmatrix} = \begin{bmatrix}\bar{y}_{1\cdot} \\ \bar{y}_{2\cdot}\end{bmatrix} ,$$

it is easily seen that in the general case (9) is

$$\hat{\underset{\sim}{\mu}} = \left\{\hat{\mu}_i\right\}_{i=1}^{i=a} = \left\{\bar{y}_{i\cdot}\right\}_{i=1}^{i=a} ;$$

i.e.,

$$\hat{\mu}_i = \bar{y}_{i\cdot} \ . \tag{10}$$

This is not unexpected: that in randomized complete blocks with blocks random, the GLS estimator of the i'th treatment (row) mean μ_i is the sample n $\bar{y}_{i\cdot}$ for that row.

Sampling variances follow directly from applying (5) and (6) to (10):

$$v(\hat{\mu}_i) = v(\bar{y}_{i\cdot}) = (\sigma_e^2 + \sigma_\beta^2)/b$$

and

$$\text{cov}(\hat{\mu}_i,\hat{\mu}_{i'}) = \text{cov}(\bar{y}_{i\cdot},\bar{y}_{i'\cdot}) = \sigma_\beta^2/b \quad \text{for } i \neq i' \ .$$

The preceding results for balanced data are familiar. We now extend them to unbalanced data

3. Unbalanced Data

a. The dispersion matrix V

In having more than 1 observation per cell the covariance structure of (5) and (6) is now

$$
\begin{aligned}
v(y_{ijk}) &= \sigma_e^2 + \sigma_\beta^2, \\
\text{cov}(y_{ijk}, y_{ijk'}) &= \sigma_\beta^2 \quad \text{for } k \neq k', \\
\text{cov}(y_{ijk}, y_{i'jk'}) &= \sigma_\beta^2 \quad \text{for } i \neq i',\ k = 1,\cdots,n_{ij} \text{ and } k' = 1,\cdots,n_{i'j}
\end{aligned}
\tag{11}
$$

and

$$\text{cov}(y_{ijk}, y_{i'j'k'}) = 0 \quad \text{for } j \neq j'.$$

Example 1 (continued): Using (11) the variance-covariance matrix of the data vector $\underset{\sim}{y}$ shown following Grid 1 is

$$\underset{\sim}{V} = \text{var}(\underset{\sim}{y})$$

$$
= e\underset{\sim}{I}_{n..} + \beta \begin{bmatrix}
\underset{\sim}{J}_{n_{11}} & \cdot & \cdot & \underset{\sim}{J}_{n_{11}\times n_{21}} & \cdot & \cdot \\
\cdot & \underset{\sim}{J}_{n_{12}} & \cdot & \cdot & \underset{\sim}{J}_{n_{12}\times n_{22}} & \cdot \\
\cdot & \cdot & \underset{\sim}{J}_{n_{13}} & \cdot & \cdot & \underset{\sim}{J}_{n_{13}\times n_{23}} \\
\underset{\sim}{J}_{n_{21}\times n_{11}} & \cdot & \cdot & \underset{\sim}{J}_{n_{21}} & \cdot & \cdot \\
\cdot & \underset{\sim}{J}_{n_{22}\times n_{12}} & \cdot & \cdot & \underset{\sim}{J}_{n_{22}} & \cdot \\
\cdot & \cdot & \underset{\sim}{J}_{n_{23}\times n_{13}} & \cdot & \cdot & \underset{\sim}{J}_{n_{23}}
\end{bmatrix}
\tag{12}
$$

where each dot is a null sub-matrix of $\underset{\sim}{V}$ of appropriate order.

The form of $\underset{\sim}{V}$ merits observation: the diagonal sub-matrices are square $\underset{\sim}{J}$-matrices having orders $n_{11}, n_{12}, \cdots, n_{1b}, \cdots \cdots, n_{a1}, \cdots, n_{ab}$, respectively. And the off-diagonal sub-matrices are rectangular $\underset{\sim}{J}$- matrices of order $n_{ij} \times n_{i'j}$ for $i \neq i'$. In the example, i has only two values, 1 and 2, and so above (and below) the diagonal (of sub-matrices $\underset{\sim}{J}_{n_{ij}}$) there is only one band of off-diagonal sub-matrices $\underset{\sim}{J}_{n_{ij} \times n_{i'j}}$. In general there are a - 1 such bands.

b. Development of V^{-1}

The vector of observations $\underset{\sim}{y}$ has been defined as containing the y_{ijk} values in lexicon order, i.e., ordered by k, within j within i. Let $\underset{\sim}{P}$ be a permutation matrix such that $\underset{\sim}{P}\underset{\sim}{y}$ contains the y_{ijk} values ordered by k within i within j; i.e.,

$$\underset{\sim}{P}\underset{\sim}{y} = \left\{\left\{\left\{y_{ijk}\right\}_{k=1}^{n_{ij}}\right\}_{i=1}^{i=a}\right\}_{j=1}^{j=b} . \tag{13}$$

Then observe that if the matrix which β multiplies in $\underset{\sim}{V}$ of (12) is pre-multiplied by $\underset{\sim}{P}$ and post-multiplied by $\underset{\sim}{P}'$ the product has the form $\bigoplus_{j=1}^{3} \underset{\sim}{J}_{n_{\cdot j}}$. Hence, in general, because $\underset{\sim}{P}$ is orthogonal (as are all permutation matrices)

$$\underset{\sim}{V} = e\underset{\sim}{I} + \beta\underset{\sim}{P}'\left(\bigoplus_{j=1}^{b} \underset{\sim}{J}_{n_{\cdot j}}\right)\underset{\sim}{P} . \tag{14}$$

Define

$$\underset{\sim}{W} = (1/e)\underset{\sim}{I} + \underset{\sim}{P}'\left(\bigoplus_{j=1}^{b} \theta_j \underset{\sim}{J}_{n_{\cdot j}}\right)\underset{\sim}{P} , \tag{15}$$

for scalars θ_j for $j = 1, \cdots, b$. We derive values for the θ_j such that $\underset{\sim}{W} = \underset{\sim}{V}^{-1}$. Consider the product $\underset{\sim}{V}\underset{\sim}{W}$: by direct multiplication of (14) and (15) it is

$$\mathbf{VW} = \mathbf{I} + \mathbf{P}'\left[\bigoplus_{j=1}^{b}(e\theta_j + \beta/e + \beta n_{.j}\theta_j)\mathbf{J}_{n_{.j}}\right]\mathbf{P} \ .$$

Hence $\mathbf{VW} = \mathbf{I}$ if

$$\theta_j = \frac{-\beta/e}{e + n_{.j}\beta} \quad \text{for } j = 1,\cdots,b \ . \tag{16}$$

Thus (16) is the condition for $\mathbf{W}$ to be $\mathbf{V}^{-1}$, and so substituting (16) into (15) gives

$$\mathbf{V}^{-1} = (1/e)\left[\mathbf{I} - \beta\mathbf{P}'\left(\bigoplus_{j=1}^{b}\lambda_j\mathbf{J}_{n_{.j}}\right)\mathbf{P}\right] \tag{17}$$

for

$$\lambda_j = \frac{1}{e + n_{.j}\beta} \ . \tag{18}$$

c. Solving the GLS equations

The cell means model is based on

$$E(y_{ijk}) = \mu_i \ .$$

just as in Section 2. There, in (4), we have

$$\mathbf{X} = (\mathbf{I}_a \otimes \mathbf{1}_b) = \bigoplus_{i=1}^{a}\mathbf{1}_b \ .$$

But now, for unbalanced data, $\mathbf{X}$ is a generalization of this second form, namely

$$\mathbf{X} = \bigoplus_{i=1}^{a}\mathbf{1}_{n_{i.}} \ . \tag{19}$$

Therefore for (3), using (17) and (19)

$$\begin{aligned}\mathbf{X}'\mathbf{V}^{-1}\mathbf{X} &= \left(\bigoplus_{i=1}^{a}\mathbf{1}'_{n_{i.}}\right)(1/e)\left[\mathbf{I} - \beta\mathbf{P}'\left(\bigoplus_{j=1}^{b}\lambda_j\mathbf{J}_{n_{.j}}\right)\mathbf{P}\right]\left(\bigoplus_{i=1}^{a}\mathbf{1}_{n_{i.}}\right) \\ &= (1/e)\left[\bigoplus_{i=1}^{a}n_{i.} - \beta\mathbf{Q}'\left(\bigoplus_{j=1}^{b}\lambda_j\mathbf{J}_{n_{.j}}\right)\mathbf{Q}\right]\end{aligned} \tag{20}$$

for

$$\underset{\sim}{Q} = \underset{\sim}{P}\left(\bigoplus_{i=1}^{a} \underset{\sim}{1}_{n_{i\cdot}}\right) . \tag{21}$$

Hence, by the definition of $\underset{\sim}{P}$ given earlier,

$$\underset{\sim}{Q} = \left\{\underset{\sim}{Q}_j\right\}_{j=1}^{b} \quad \text{for} \quad \underset{\sim}{Q}_j = \bigoplus_{i=1}^{a} \underset{\sim}{1}_{n_{ij}} . \tag{22}$$

To establish the form of the product that involves $\underset{\sim}{Q}$ in (20) we first consider the example

Example (continued): Part of (20) is

$$\underset{\sim}{Q}'\left(\bigoplus_{j=1}^{3} \lambda_j \underset{\sim}{J}_{n_{\cdot j}}\right)\underset{\sim}{Q}$$

$$= \begin{bmatrix} \underset{\sim}{1}'_{n_{11}} & \cdot & \underset{\sim}{1}'_{n_{12}} & \cdot & \underset{\sim}{1}'_{n_{13}} & \cdot \\ \cdot & \underset{\sim}{1}'_{n_{21}} & \cdot & \underset{\sim}{1}'_{n_{22}} & \cdot & \underset{\sim}{1}'_{n_{23}} \end{bmatrix} \begin{bmatrix} \lambda_1 \underset{\sim}{J}_{n_{\cdot 1}} & \cdot & \cdot \\ \cdot & \lambda_2 \underset{\sim}{J}_{n_{\cdot 2}} & \cdot \\ \cdot & \cdot & \lambda_3 \underset{\sim}{J}_{n_{\cdot 3}} \end{bmatrix} \begin{bmatrix} \underset{\sim}{1}_{n_{11}} & \cdot \\ \cdot & \underset{\sim}{1}_{n_{21}} \\ \underset{\sim}{1}_{n_{12}} & \cdot \\ \cdot & \underset{\sim}{1}_{n_{22}} \\ \underset{\sim}{1}_{n_{13}} & \cdot \\ \cdot & \underset{\sim}{1}_{n_{23}} \end{bmatrix} \tag{23}$$

$$= \begin{bmatrix} \lambda_1 n_{11}^2 + \lambda_2 n_{12}^2 + \lambda_3 n_{13}^2 & \lambda_1 n_{11} n_{21} + \lambda_2 n_{12} n_{22} + \lambda_3 n_{13} n_{23} \\ \lambda_1 n_{11} n_{21} + \lambda_2 n_{12} n_{22} + \lambda_3 n_{13} n_{23} & \lambda_1 n_{21}^2 + \lambda_2 n_{22}^2 + \lambda_3 n_{23}^2 \end{bmatrix} \tag{24}$$

$$= \lambda_1 \begin{bmatrix} n_{11} \\ n_{21} \end{bmatrix} [n_{11} \quad n_{21}] + \lambda_2 \begin{bmatrix} n_{12} \\ n_{22} \end{bmatrix} [n_{12} \quad n_{22}] + \lambda_3 \begin{bmatrix} n_{13} \\ n_{23} \end{bmatrix} [n_{13} \quad n_{23}] . \tag{23}$$

(24) comes from (23) by direct multiplication: each element of (24) is a quadratic (or bilinear) form involving $+\lambda_j \underset{\sim}{J}_{n_{.j}}$. And (25) comes from (24) by observation.

Generalization of (25) is clear: define

$$\underset{\sim}{c}_j = [n_{1j} \quad n_{2j} \quad \cdots \quad n_{aj}]' , \tag{26}$$

the column vector of the numbers of observations in the j'th column of the data. Then

$$\underset{\sim}{Q}'\left(\bigoplus_{j=1}^{b} \lambda_j \underset{\sim}{J}_{n_{.j}}\right)\underset{\sim}{Q} = \sum_{j=1}^{b} \lambda_j \underset{\sim}{c}_j \underset{\sim}{c}_j' . \tag{27}$$

Therefore in (20)

$$\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X} = (1/e)\left[\bigoplus_{i=1}^{a} n_{i.} - \beta \sum_{j=1}^{b} \frac{1}{e + n_{.j}\beta} \underset{\sim}{c}_j \underset{\sim}{c}_j'\right] . \tag{28}$$

Similarly,

$$\begin{aligned} \underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{y} &= \left(\bigoplus_{i=1}^{a} \underset{\sim}{1}'_{n_{i.}}\right)(1/e)\left[I - \beta \underset{\sim}{P}'\left(\bigoplus_{j=1}^{b} \lambda_j \underset{\sim}{J}_{n_{.j}}\right)\underset{\sim}{P}\right]\underset{\sim}{y} \\ &= (1/e)\left[\left\{y_{i..}\right\}_{i=1}^{i=a} - \beta \underset{\sim}{Q}'\left(\bigoplus_{j=1}^{b} \lambda_j \underset{\sim}{J}_{n_{.j}}\right)\left\{\left\{\left\{y_{ijk}\right\}_{k=1}^{n_{ij}}\right\}_{i=1}^{i=a}\right\}_{j=1}^{j=b}\right] \\ &= (1/e)\left[\left\{y_{i..}\right\}_{i=1}^{i=a} - \beta \underset{\sim}{Q}'\left\{\lambda_j y_{.j.} \underset{\sim}{1}_{n_{.j}}\right\}_{j=1}^{j=b}\right] . \end{aligned}$$

Hence, from the nature of $\underset{\sim}{Q}$ in (22), exemplified in (23),

$$\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{y} = (1/e)\left[\left\{y_{i..}\right\}_{i=1}^{i=a} - \beta\left\{\sum_{j=1}^{b} n_{ij}\lambda_j y_{.j.}\right\}_{i=1}^{i=a}\right] .$$

Thus the solution to the normal equations is

$$\left\{\hat{\mu}_i\right\}_{i=1}^{i=a} = (\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{y}$$

$$= \left[\bigoplus_{i=1}^{a} n_{i\cdot} - \beta \sum_{j=1}^{b} \frac{1}{e + n_{\cdot j}\beta}\, \underset{\sim}{c}_j \underset{\sim}{c}_j'\right]^{-1} \left\{y_{i\cdot\cdot} - \beta \sum_{j=1}^{b} \frac{n_{ij}y_{\cdot j\cdot}}{e + n_{\cdot j}\beta}\right\}_{i=1}^{i=a} \tag{29}$$

where

$$\underset{\sim}{c}_j = \left\{n_{ij}\right\}_{i=1}^{i=a}, \quad e \equiv \sigma_e^2 \quad \text{and} \quad \beta \equiv \sigma_\beta^2 \ .$$

Unfortunately, the matrix inverse required in (29) seems to have no explicit form. It does for special cases as shown in Section 4.

Since $\text{var}(\underset{\sim}{y}) = \underset{\sim}{V}$, it is clear that $\text{var}(\hat{\mu}) = (\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})^{-1}$, as is well known. Hence, from (28)

$$\text{var}(\hat{\underset{\sim}{\mu}}) = e\left[\bigoplus_{i=1}^{a} n_{i\cdot} - \beta \sum_{j=1}^{b} \frac{1}{e + n_{\cdot j}\beta}\, \underset{\sim}{c}_j \underset{\sim}{c}_j'\right]^{-1} . \tag{30}$$

Some cells empty

The general result (29) has been developed on the implicit assumption that all cells are filled, i.e., that all $n_{ij} > 0$. But, in fact, this assumption has not been used and is not necessary: the crucial feature of the development of (29) is (14), which holds true even for some n_{ij} being zero. Thus (29) does not depend on $n_{ij} > 0$, and so is applicable both for all-cells-filled data and for some-cells-empty data.

4. Three Special Cases

We show details of three special cases of the 2-way crossed classification: randomized complete blocks, split plots and balanced incomplete blocks, the first two of which are commented on by Steinhorst (1982).

a. Randomized Complete Blocks (RCB)

As presented in Section 2, data from an RCB experiment are the special case of unbalanced data with all $n_{ij} = 1$. This reduces (29) to

$$\left\{\hat{\mu}_i\right\}_{i=1}^{a} = \left(b\underset{\sim}{I}_a - \frac{\beta b}{e + a\beta}\underset{\sim}{1}_a\underset{\sim}{1}_a'\right)^{-1}\left\{y_{i\cdot} - \frac{\beta y_{\cdot\cdot}}{e + a\beta}\right\}_{i=1}^{i=a} \tag{31}$$

$$= (1/b)\left(\underset{\sim}{I}_a - \frac{\beta}{e + a\beta}\underset{\sim}{J}_a\right)^{-1}\left\{y_{i\cdot} - \frac{\beta y_{\cdot\cdot}}{e + a\beta}\right\}_{i=1}^{i=a}$$

$$= (1/b)[\underset{\sim}{I}_a + (\beta/e)\underset{\sim}{J}_a]\left\{y_{i\cdot} - \frac{\beta}{e + a\beta}y_{\cdot\cdot}\right\}_{i=1}^{i=a}$$

$$= (1/b)\left\{y_{i\cdot} + (\beta/e)y_{\cdot\cdot} - [\beta/(e + a\beta)](1 + a\beta/e)y_{\cdot\cdot}\right\}_{i=1}^{i=a}$$

$$= \left\{\bar{y}_{i\cdot}\right\}_{i=1}^{i=a} ,$$

i.e., $\hat{\mu}_i = \bar{y}_{i\cdot}$, precisely as in (10). And, from (30) and its occurrence in (31),

$$\text{var}(\underset{\sim}{\hat{\mu}}) = e\left(b\underset{\sim}{I}_a - \frac{\beta b}{e + a\beta}\underset{\sim}{J}_a\right)^{-1} = (e/b)(\underset{\sim}{I}_a + \beta\underset{\sim}{J}_a/e)$$

so giving

$$v(\hat{\mu}_i) = v(\bar{y}_{i\cdot}) = (e/b)(1 + \beta/e) = (\sigma_\beta^2 + \sigma_e^2)/b$$

and

$$\text{cov}(\hat{\mu}_i,\hat{\mu}_{i'}) = \text{cov}(\bar{y}_{i\cdot},\bar{y}_{i'\cdot}) = (e/b)\beta/e = \sigma_\beta^2/b ,$$

b. Split plots in randomized complete blocks

A traditional (over-parameterized) model for a split-plot experiment in a randomized complete blocks design is to define

$$E(y_{ijk}) = \mu + \alpha_i + \gamma_k + (\alpha\gamma)_{ik} + \beta_j + (\alpha\beta)_{ij} . \tag{32}$$

A cell means representation of this is the 2-way crossed classification

$$E(y_{ijk}) = \mu_{ik} \tag{33}$$

with b observations in each (i,k) cell, and with the following variance-covariance structure:

$$\begin{aligned}
v(y_{ijk}) &= \sigma^2_\beta + \sigma^2_{\alpha\beta} + \sigma^2_e \\
\mathrm{cov}(y_{ijk}, y_{ijk'}) &= \sigma^2_\beta + \sigma^2_{\alpha\beta} \quad \text{for } k \neq k' \\
\mathrm{cov}(y_{ijk}, y_{i'jk'}) &= \sigma^2_\beta \quad \text{for } i \neq i' \text{ and } k \neq k' \\
\mathrm{cov}(y_{ijk}, y_{i'j'k'}) &= 0 \quad \text{for } j \neq j' .
\end{aligned}$$

On arraying the observations in lexicon order as

$$\underset{\sim}{y} = \left\{\left\{\left\{y_{ijk}\right\}_{i=1}^{i=a}\right\}_{j=1}^{j=b}\right\}_{k=1}^{k=c} \tag{34}$$

the variance-covariance matrix of $\underset{\sim}{y}$ can be written as

$$\mathrm{var}(\underset{\sim}{y}) = \underset{\sim}{V} = e\underset{\sim}{I}_{abc} + \beta(\underset{\sim}{J}_a \otimes \underset{\sim}{I}_b \otimes \underset{\sim}{J}_c) + \phi(\underset{\sim}{I}_a \otimes \underset{\sim}{I}_b \otimes \underset{\sim}{J}_c) . \tag{35}$$

using the notation

$$e = \sigma^2_e, \quad \beta = \sigma^2_\beta \quad \text{and} \quad \phi = \sigma^2_{\alpha\beta} .$$

Then it will be found, similar to the methods of Searle and Henderson (1979), that for $\underset{\sim}{V}$ of (35)

$$\underset{\sim}{V}^{-1} = (1/e)\underset{\sim}{I}_{abc} + t(\underset{\sim}{J}_a \otimes \underset{\sim}{I}_b \otimes \underset{\sim}{J}_c) + u(\underset{\sim}{I}_a \otimes \underset{\sim}{I}_b \otimes \underset{\sim}{J}_c) \tag{36}$$

where

$$t = \frac{-\beta}{(e + c\phi)(e + ac\beta + c\phi)} \quad \text{and} \quad u = \frac{-\phi}{e(e + c\phi)} . \tag{37}$$

Further, with the y_{ijk}'s arrayed in $\underset{\sim}{y}$ in lexicon order of (34)

$$E(\underset{\sim}{y}) = (\underset{\sim}{I}_a \otimes \underset{\sim}{1}_b \otimes \underset{\sim}{I}_c)\left\{\left\{\mu_{ij}\right\}_{i=1}^{i=a}\right\}_{k=1}^{k=c}$$

so that

$$\underset{\sim}{X} = (\underset{\sim}{I}_a \otimes \underset{\sim}{1}_b \otimes \underset{\sim}{I}_c) \; .$$

Hence, pre-multiplying each term of (36) by $\underset{\sim}{X}'$ and post-multiplying by $\underset{\sim}{X}$ gives

$$\begin{aligned}\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X} &= (1/e)(\underset{\sim}{I}_a \otimes b \otimes \underset{\sim}{I}_c) + t(\underset{\sim}{J}_a \otimes b \otimes \underset{\sim}{J}_c) + u(\underset{\sim}{I}_a \otimes b \otimes \underset{\sim}{J}_c) \\ &= b[(1/e)(\underset{\sim}{I}_a \otimes \underset{\sim}{I}_c) + t(\underset{\sim}{J}_a \otimes \underset{\sim}{J}_c) + u(\underset{\sim}{I}_a \otimes \underset{\sim}{J}_c)] \; . \qquad (38)\end{aligned}$$

And then it will be found that

$$(\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})^{-1} = (1/b)[e(\underset{\sim}{I}_a \otimes \underset{\sim}{I}_c) + \beta(\underset{\sim}{J}_a \otimes \underset{\sim}{J}_c) + \phi(\underset{\sim}{I}_a \otimes \underset{\sim}{J}_c)] \; . \qquad (39)$$

Similarly

$$\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{y} = [(1/e)(\underset{\sim}{I}_a \otimes \underset{\sim}{1}_b' \otimes \underset{\sim}{I}_c) + t(\underset{\sim}{J}_a \otimes \underset{\sim}{1}_b' \otimes \underset{\sim}{J}_c) + u(\underset{\sim}{I}_a \otimes \underset{\sim}{1}_b' \otimes \underset{\sim}{J}_c)]\underset{\sim}{y} \; . \qquad (40)$$

Observe that

$$(\underset{\sim}{I}_a \otimes \underset{\sim}{1}_b' \otimes \underset{\sim}{I}_c) = (\underset{\sim}{I}_a \otimes 1 \otimes \underset{\sim}{I}_c)(\underset{\sim}{I}_a \otimes \underset{\sim}{1}_b' \otimes \underset{\sim}{I}_c) = (\underset{\sim}{I}_a \otimes \underset{\sim}{I}_c)(\underset{\sim}{I}_a \otimes \underset{\sim}{1}_b' \otimes \underset{\sim}{I}_c).$$

Applying this principle to each term in (40) gives

$$\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{y} = [(1/e)(\underset{\sim}{I}_a \otimes \underset{\sim}{I}_c) + t(\underset{\sim}{J}_a \otimes \underset{\sim}{J}_c) + u(\underset{\sim}{I}_a \otimes \underset{\sim}{J}_c)](\underset{\sim}{I}_a \otimes \underset{\sim}{1}_b' \otimes \underset{\sim}{I}_c)\underset{\sim}{y}$$

and on comparison with (38) this can be written as

$$\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{y} = (1/b)\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X}(\underset{\sim}{I}_a \otimes \underset{\sim}{1}_b' \otimes \underset{\sim}{I}_c)\underset{\sim}{y} \; .$$

Hence

$$\begin{aligned}\hat{\underset{\sim}{\mu}} &= (\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{y} \\ &= (1/b)(\underset{\sim}{I}_a \otimes \underset{\sim}{1}_b' \otimes \underset{\sim}{I}_c)\underset{\sim}{y}\end{aligned}$$

giving

$$\hat{\mu}_{ik} = \bar{y}_{i \cdot k} \; . \qquad (41)$$

And this is, of course, precisely the estimator of $\mu + \alpha_i + \gamma_k + (\alpha\gamma)_{ik}$ obtained in the overparameterized model — as one would anticipate.

From (39), using $\text{var}(\hat{\mu}) = (\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})^{-1}$, we then get anticipated results for sampling variances:

$$v(\hat{\mu}_{i\cdot k}) = v(\bar{y}_{i\cdot k}) = (\sigma_e^2 + \sigma_\beta^2 + \sigma_\phi^2)/b \ ,$$

$$\text{cov}(\hat{\mu}_{i\cdot k},\hat{\mu}_{i\cdot k'}) = \text{cov}(\bar{y}_{i\cdot k},\bar{y}_{i\cdot k'}) = (\sigma_\beta^2 + \sigma_\phi^2)/b, \text{ for } k' \neq k$$

and

$$\text{cov}(\hat{\mu}_{i\cdot k},\hat{\mu}_{i'\cdot k'}) = \text{cov}(\bar{y}_{i\cdot k},\bar{y}_{i'\cdot k'}) = \sigma_\beta^2/b \quad \text{for } i \neq i' \ .$$

c. Balanced incomplete blocks (BIB)

Data from a balanced incomplete blocks experiment can be arrayed in the grid of a 2-way crossed classification with values of n_{ij} being 0 and 1 in a patterned manner determined by the nature of the experiment.

Example 3: Consider the case of 4 treatments ($a = 4$) used in a BIB experiment of 6 blocks ($b = 6$) with 2 treatments in each block. The pattern of n_{ij}-values can be arrayed as in Grid 2, where a dash represents no observation.

Grid 2

Treatment	Block						
	1	2	3	4	5	6	$n_{i\cdot} = r$
I	1	1	1	—	—	—	3
II	1	—	—	1	1	—	3
III	—	1	—	1	—	1	3
IV	—	—	1	—	1	1	3
$n_{\cdot j} = k$	2	2	2	2	2	2	$12 = n_{\cdot\cdot} = ar = kb$

The general description of a BIB experiment customarily involves the following characteristics:

b = number of blocks

k = number of different treatments used in each block

$a = t$ = number of treatments

r = number of blocks that contain each particular treatment

λ = number of times each treatment pair occurs in the same block.

Although t is the traditional symbol for the number of treatments, we use a here for consistency with our general description of the 2-way classification. In terms of that description we can also note the following relationships for both the general case and the example.

$$\begin{array}{lll} a = 4 & n_{i\cdot} = r = 3 & \lambda = 1 \\ b = 6 & n_{\cdot j} = k = 2 & \\ & n_{\cdot\cdot} = ar = bk = 6. & \end{array}$$

Furthermore, there is the usual equality for BIB experiments, that

$$\lambda(a - 1) = r(k - 1) \ . \tag{42}$$

To simplify (29) first note that any cell containing data has only one observation (BIB designs with more than one can be considered, but are not dealt with here), and so we denote it by y_{ij}. Then for (29) we have

$$\left\{\hat{\mu}_i\right\}_{i=1}^{i=a} = \left[r\underset{\sim}{I}_a - \frac{\beta}{e + k\beta} \sum_{j=1}^{b} \underset{\sim}{c}_j\underset{\sim}{c}_j'\right]^{-1} \left\{y_{i\cdot} - \frac{\beta}{e + k\beta} \sum_{j=1}^{b} n_{ij}y_{\cdot j}\right\}_{i=1}^{i=a} . \tag{43}$$

This requires simplifying two summation terms. The first is done with assistance of the example.

Example 3 (continued): Using the columns of unities and zeros in Grid 2 as the columns $\underset{\sim}{c}_j$,

$$\sum_{j=1}^{b} \underset{\sim}{c}_j \underset{\sim}{c}_j' = \begin{bmatrix} 1 1 \cdot \cdot \\ 1 1 \cdot \cdot \\ \cdot \cdot \cdot \cdot \\ \cdot \cdot \cdot \cdot \end{bmatrix} + \begin{bmatrix} 1 \cdot 1 \cdot \\ \cdot \cdot \cdot \cdot \\ 1 \cdot 1 \cdot \\ \cdot \cdot \cdot \cdot \end{bmatrix} + \begin{bmatrix} 1 \cdot \cdot 1 \\ \cdot \cdot \cdot \cdot \\ \cdot \cdot \cdot \cdot \\ 1 \cdot \cdot 1 \end{bmatrix} + \begin{bmatrix} \cdot \cdot \cdot \cdot \\ \cdot 1 1 \cdot \\ \cdot 1 1 \cdot \\ \cdot \cdot \cdot \cdot \end{bmatrix} + \begin{bmatrix} \cdot \cdot \cdot \cdot \\ \cdot 1 \cdot 1 \\ \cdot \cdot \cdot \cdot \\ \cdot 1 \cdot 1 \end{bmatrix} + \begin{bmatrix} \cdot \cdot \cdot \cdot \\ \cdot \cdot \cdot \cdot \\ \cdot \cdot 1 1 \\ \cdot \cdot 1 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} = (3 - 1)\underset{\sim}{I}_4 + \underset{\sim}{J}_4 \; .$$

Generalization is that

$$\sum_{j=1}^{b} \underset{\sim}{c}_j \underset{\sim}{c}_j' = (r - \lambda)\underset{\sim}{I}_a + \lambda \underset{\sim}{J}_a \; . \tag{44}$$

The second summation for (43) is

$$\sum_{j=1}^{b} n_{ij} y_{\cdot j} = \sum_{j=1}^{b} n_{ij} k\bar{y}_{\cdot j} = kr\left(\sum_{j=1}^{b} n_{ij} \bar{y}_{\cdot j}\right)/r = kr\bar{y}_{i(j)} \tag{45}$$

where

$$\bar{y}_{i(j)} = \sum_{j=1}^{b} n_{ij} \bar{y}_{\cdot j \cdot}/r = \text{mean of block means } \bar{y}_{\cdot j \cdot} \text{ for the blocks that contain treatment } i.$$

Substituting (44) and (45) into (43) gives

$$\left\{\hat{\mu}_i\right\}_{i=1}^{i=a} = \left[r\underset{\sim}{I}_a - \frac{\beta(r - \lambda)}{e + k\beta}\underset{\sim}{I}_a - \frac{\beta\lambda}{e + k\beta}\underset{\sim}{J}_a\right]^{-1}\left\{y_{i\cdot} - \frac{\beta kr}{e + k\beta}\bar{y}_{i(j)}\right\}_{i=1}^{i=a}$$

$$= (e + k\beta)\Big([re + (rk - r + \lambda)\beta]\underset{\sim}{I}_a - \beta\lambda\underset{\sim}{J}_a\Big)^{-1}\left\{y_{i\cdot} - \frac{kr\beta}{e + k\beta}\bar{y}_{i(j)}\right\}_{i=1}^{i=a} .$$

But (42) gives $rk - r + \lambda = \lambda a$. Therefore

$$\left\{\hat{\mu}_i\right\}_{i=1}^{i=a} = (e + k\beta)[re + \lambda a\beta)\underset{\sim}{I}_a - \beta\lambda\underset{\sim}{J}_a]^{-1}\left\{y_{i\cdot} - \frac{kr\beta}{e + k\beta}\,\bar{y}_{i(j)}\right\}_{i=1}^{i=a}$$

$$= \frac{e + k\beta}{re + \lambda a\beta}\left(\underset{\sim}{I}_a + \frac{\lambda\beta}{re}\underset{\sim}{J}_a\right)\left\{y_{i\cdot} - \frac{kr\beta}{e + k\beta}\,\bar{y}_{i(j)}\right\}_{i=1}^{i=a} .$$

Hence

$$\hat{\mu}_i = \frac{e + k\beta}{re + \lambda a\beta}\left[y_{i\cdot} + \frac{\lambda\beta}{re}\,y_{\cdot\cdot} - \frac{kr\beta}{e + k\beta}\,\bar{y}_{i(j)} - \frac{\lambda\beta}{re}\,\frac{kr\beta}{e + k\beta}\sum_{i=1}^{a}\bar{y}_{i(j)}\right] .$$

But from (45)

$$\sum_{i=1}^{a}\bar{y}_{i(j)} = \sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}\bar{y}_{\cdot j}/r = \sum_{j=1}^{b} n_{\cdot j}\bar{y}_{\cdot j}/r = y_{\cdot\cdot}/r .$$

Therefore

$$\hat{\mu}_i = \frac{e + k\beta}{re + \lambda a\beta}\left[y_{i\cdot} - \frac{kr\beta}{e + k\beta}\,\bar{y}_{i(j)} + \frac{\lambda\beta}{re}\left(1 - \frac{k\beta}{e + k\beta}\right)y_{\cdot\cdot}\right]$$

$$= \frac{r(e + k\beta)}{re + \lambda a\beta}\left[\bar{y}_{i\cdot} - \frac{k\beta}{e + k\beta}\,\bar{y}_{i(j)} + \frac{\lambda a\beta}{r(e + k\beta)}\,\bar{y}_{\cdot\cdot}\right] . \qquad (46)$$

As shown in the appendix, this result is consistent with results given in Scheffé (1959).

Furthermore, from (30), using intermediate steps in the derivation of $\hat{\mu}_i$,

$$\text{var}(\mu) = e\left[r\underset{\sim}{I}_a - \frac{\beta(r - \lambda)}{e + k\beta}\,\underset{\sim}{I}_a - \frac{\beta\lambda}{e + k\beta}\,\underset{\sim}{J}_a\right]^{-1}$$

$$= \frac{e(e + k\beta)}{re + \lambda a\beta}\left(\underset{\sim}{I}_a + \frac{\lambda\beta}{re}\,\underset{\sim}{J}_a\right)$$

so that

$$v(\hat{\mu}_i) = \frac{(e + k\beta)(re + \lambda\beta)}{r(re + \lambda a\beta)} \qquad (47)$$

and

$$\text{cov}(\hat{\mu}_i,\hat{\mu}_{i'}) = \frac{\lambda\beta(e + k\beta)}{r(re + \lambda a\beta)} \quad \text{for } i \neq i' .$$

These results are sometimes written in terms of

$$\rho = \sigma_\beta^2/\sigma_e^2:$$

$$v(\hat{\mu}_i) = \frac{(1 + k\rho)(1 + \lambda\rho/r)}{1 + a\lambda\rho/r}\,\sigma_e^2$$

and

$$\mathrm{cov}(\hat{\mu}_i\hat{\mu}_{i'}) = \frac{(\lambda\rho/r)(1 + k\rho)}{1 + a\lambda\rho/r}\,\sigma_e^2 \quad \text{for } i \neq i' \,.$$

Finally, it can be noted in passing that when $\lambda = r = b$ and $k = a$, a BIB becomes an RCB whereupon (46) reduces to

$$\hat{\mu}_i = \frac{b(e + a\beta)}{b(e + a\beta)}\left[\bar{y}_{i\cdot} - \frac{a\beta}{e + a\beta}\,\bar{y}_{\cdot\cdot} + \frac{ba\beta}{b(e + a\beta)}\,\bar{y}_{\cdot\cdot}\right] = \bar{y}_{i\cdot}\ ,$$

as is to be expected.

5. Estimating Cell Means

a. Without within-cell covariance

Suppose, despite the within-column covariance represented by σ_β^2 in the preceding development, that there was interest in estimating cell means μ_{ij} with

$$E(y_{ijk}) = \mu_{ij}\ .$$

Then

$$\underset{\sim}{X} = \bigoplus_{i=1}^{a}\left(\bigoplus_{j=1}^{b}\underset{\sim}{1}_{n_{ij}}\right) . \tag{48}$$

Using (48) and (17) we therefore have, similar to (20),

$$\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X} = \bigoplus_{i=1}^{a}\left(\bigoplus_{j=1}^{b}\underset{\sim}{1}'_{n_{ij}}\right)(1/e)\left[\underset{\sim}{I} - \beta\underset{\sim}{P}'\left(\bigoplus_{j=1}^{b}\lambda_j\underset{\sim}{J}_{n_{\cdot j}}\right)\underset{\sim}{P}\right]\bigoplus_{i=1}^{a}\left(\bigoplus_{j=1}^{b}\underset{\sim}{1}_{n_{ij}}\right)$$

$$= (1/e)\left[\bigoplus_{i=1}^{a}\ \bigoplus_{j=1}^{b} n_{ij} - \beta\underset{\sim}{Q}'\left(\bigoplus_{j=1}^{b}\lambda_j\underset{\sim}{J}_{n_{\cdot j}}\right)\underset{\sim}{Q}\right] \tag{49}$$

for

$$\underset{\sim}{Q} = \underset{\sim}{P}\underset{\sim}{X} = \underset{\sim}{P}\bigoplus_{i=1}^{a}\bigoplus_{j=1}^{b}\underset{\sim}{1}_{n_{ij}}$$

where $\underset{\sim}{P}$ is the permutation matrix defined in (13). $\underset{\sim}{P}$ has order $n_{..}$. We now define another permutation matrix, $\underset{\sim}{T}$, of order ab such that

$$\underset{\sim}{T}\left[\left\{\left\{y_{ij.}\right\}_{j=1}^{j=b}\right\}_{i=1}^{i=a}\right] = \left\{\left\{y_{ij.}\right\}_{i=1}^{i=a}\right\}_{j=1}^{j=b} .$$

Then

$$\underset{\sim}{Q} = \underset{\sim}{P}\underset{\sim}{X}\underset{\sim}{T}'\underset{\sim}{T} \quad \text{where, for } \underset{\sim}{X} \text{ of (48),} \quad \underset{\sim}{P}\underset{\sim}{X}\underset{\sim}{T}' = \bigoplus_{j=1}^{b}\left(\bigoplus_{i=1}^{a}\underset{\sim}{1}_{n_{ij}}\right) .$$

Hence in (49)

$$\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X} = (1/e)\left[\bigoplus_{i=1}^{a}\bigoplus_{j=1}^{b}n_{ij} - \beta\underset{\sim}{T}'(\underset{\sim}{P}\underset{\sim}{X}\underset{\sim}{T}')'\left(\bigoplus_{j=1}^{b}\lambda_j\underset{\sim}{J}_{n_{.j}}\right)(\underset{\sim}{P}\underset{\sim}{X}\underset{\sim}{T}')\underset{\sim}{T}\right] \tag{50}$$

$$= (1/e)\left(\bigoplus_{i=1}^{a}\bigoplus_{j=1}^{b}n_{ij} - \beta\underset{\sim}{T}'\bigoplus_{j=1}^{b}\left[\left(\bigoplus_{i=1}^{a}\underset{\sim}{1}'_{n_{ij}}\right)\lambda_j\underset{\sim}{J}_{n_{.j}}\left(\bigoplus_{i=1}^{a}\underset{\sim}{1}_{n_{ij}}\right)\right]\underset{\sim}{T}\right)$$

$$= (1/e)\left[\underset{\sim}{T}'\underset{\sim}{T}\left(\bigoplus_{i=1}^{a}\bigoplus_{j=1}^{b}n_{ij}\right)\underset{\sim}{T}'\underset{\sim}{T} - \beta\underset{\sim}{T}'\bigoplus_{j=1}^{b}\lambda_j\left(\left\{n_{ij}n_{i'j}\right\}_{i,i'=1}^{i,i'=a}\right)\underset{\sim}{T}\right]$$

$$= (1/e)\underset{\sim}{T}'\left[\bigoplus_{j=1}^{b}\bigoplus_{i=1}^{a}n_{ij} - \beta\bigoplus_{j=1}^{b}\lambda_j\underset{\sim}{c}_j\underset{\sim}{c}'_j\right]\underset{\sim}{T}$$

$$= (1/e)\underset{\sim}{T}'\bigoplus_{j=1}^{b}\left(\bigoplus_{i=1}^{a}n_{ij} - \beta\lambda_j\underset{\sim}{c}_j\underset{\sim}{c}'_j\right)\underset{\sim}{T} . \tag{51}$$

Example 1 (continued): The central portion of the second term in (50) is

$$(\underset{\sim}{P}\underset{\sim}{X}\underset{\sim}{T}')'\left(\bigoplus_{j=1}^{b} \lambda_j \underset{\sim}{J}_{n_{.j}}\right)\underset{\sim}{P}\underset{\sim}{X}\underset{\sim}{T}'$$

$$= \begin{bmatrix} \underset{\sim}{1}'_{n_{11}} & & & & & \\ & \underset{\sim}{1}'_{n_{21}} & & & \underset{\sim}{0} & \\ & & \underset{\sim}{1}'_{n_{12}} & & & \\ & & & \underset{\sim}{1}'_{n_{22}} & & \\ & \underset{\sim}{0} & & & \underset{\sim}{1}'_{n_{13}} & \\ & & & & & \underset{\sim}{1}'_{n_{23}} \end{bmatrix} \begin{bmatrix} \lambda_1 \underset{\sim}{J}_{n_{.1}} & \underset{\sim}{0} \\ \underset{\sim}{0} & \lambda_2 \underset{\sim}{J}_{n_{.2}} \end{bmatrix} \begin{bmatrix} \underset{\sim}{1}_{n_{11}} & & & & & \\ & \underset{\sim}{1}_{n_{21}} & & & \underset{\sim}{0} & \\ & & \underset{\sim}{1}_{n_{12}} & & & \\ & & & \underset{\sim}{1}_{n_{22}} & & \\ & \underset{\sim}{0} & & & \underset{\sim}{1}_{n_{13}} & \\ & & & & & \underset{\sim}{1}_{n_{23}} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 n_{11}^2 & \lambda_1 n_{11} n_{21} & & & & \\ \lambda_1 n_{21} n_{11} & \lambda_1 n_{12}^2 & \underset{\sim}{0} & & \underset{\sim}{0} & \\ & & \lambda_2 n_{12}^2 & \lambda_2 n_{12} n_{22} & & \\ & \underset{\sim}{0} & \lambda_2 n_{22} n_{12} & \lambda_2 n_{22}^2 & \underset{\sim}{0} & \\ & & & & \lambda_3 n_{13}^2 & \lambda_3 n_{13} n_{23} \\ & \underset{\sim}{0} & & \underset{\sim}{0} & \lambda_3 n_{23} n_{13} & \lambda_3 n_{23}^2 \end{bmatrix} = \bigoplus_{j=1}^{3} \lambda_j \underset{\sim}{c}_j \underset{\sim}{c}'_j .$$

We seek the inverse of (51). First, as a special, well-known case of (17) of Searle (1982, p. 261), note that

$$\underset{\sim}{D} - \theta \underset{\sim}{t}\underset{\sim}{t}' = \underset{\sim}{D}^{-1} + \frac{\theta \underset{\sim}{D}^{-1} \underset{\sim}{t}\underset{\sim}{t}' \underset{\sim}{D}^{-1}}{1 - \theta \underset{\sim}{t}' \underset{\sim}{D}^{-1} \underset{\sim}{t}} . \tag{52}$$

Then with

$$\bigoplus_{i=1}^{a} (1/n_{ij}) \underset{\sim}{c}_j = \left[\bigoplus_{i=1}^{a} (1/n_{ij})\right] \{n_{ij}\}_{i=1}^{i=a} = \underset{\sim}{1}_a$$

an application of (52) gives

$$\left(\bigoplus_{i=1}^{a} n_{ij} - \beta\lambda_j \underset{\sim}{c}_j \underset{\sim}{c}_j'\right)^{-1} = \bigoplus_{i=1}^{a}(1/n_{ij}) + \frac{\beta\lambda_j \underset{\sim}{1}_a \underset{\sim}{1}_a'}{1 - \beta\lambda_j \underset{\sim}{1}'\underset{\sim}{c}_j}$$

$$= \bigoplus_{i=1}^{a} \frac{1}{n_{ij}} + \frac{\beta \underset{\sim}{J}_a}{(e + n_{.j}\beta)\left(1 - \frac{\beta n_{.j}}{e + n_{.j}\beta}\right)}$$

$$= \bigoplus_{i=1}^{a} \frac{1}{n_{ij}} + \frac{\beta}{e} \underset{\sim}{J}_a \ .$$

Hence the inverse of (51) is

$$(\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})^{-1} = e\underset{\sim}{T}^{-1} \bigoplus_{j=1}^{b}\left(\bigoplus_{i=1}^{a} n_{ij} - \beta\lambda_j \underset{\sim}{c}_j \underset{\sim}{c}_j'\right)^{-1} \underset{\sim}{T}'^{-1}$$

$$= \underset{\sim}{T}' \bigoplus_{j=1}^{b}\left(e \bigoplus_{i=1}^{a} \frac{1}{n_{ij}} + \beta \underset{\sim}{J}_a\right)\underset{\sim}{T} \ . \tag{53}$$

Similarly

$$\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{y} = \bigoplus_{i=1}^{a}\left(\bigoplus_{j=1}^{b} \underset{\sim}{1}'_{n_{ij}}\right)(1/e)\left[\underset{\sim}{I} - \beta\underset{\sim}{P}'\left(\bigoplus_{j=1}^{b} \lambda_j \underset{\sim}{J}_{n_{.j}}\right)\underset{\sim}{P}\right]\underset{\sim}{y}$$

$$= (1/e)\left[\left\{\left\{y_{ij.}\right\}_{j=1}^{j=b}\right\}_{i=1}^{i=a} - \beta\underset{\sim}{T}'(\underset{\sim}{P}\underset{\sim}{X}\underset{\sim}{T}')'\left(\bigoplus_{j=1}^{b} \lambda_j \underset{\sim}{J}_{n_{.j}}\right)\underset{\sim}{P}\underset{\sim}{y}\right]$$

$$= (1/e)\underset{\sim}{T}'\left\{\left\{y_{ij.} - \beta n_{ij}\lambda_j y_{.j.}\right\}_{i=1}^{i=a}\right\}_{j=1}^{j=b} \tag{54}$$

Therefore, on using (53) and (54),

$$\underset{\sim}{T}'\underset{\sim}{T}\left\{\left\{\hat{\mu}_{ij}\right\}_{j=1}^{j=b}\right\}_{i=1}^{i=a} = (\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{y}$$

$$= \underset{\sim}{T}' \bigoplus_{j=1}^{b}\left(e \bigoplus_{i=1}^{a} \frac{1}{n_{ij}} + \beta \underset{\sim}{J}_a\right)\underset{\sim}{T}(1/e)\underset{\sim}{T}'\left\{\left\{y_{ij.} - \beta n_{ij}\lambda_j y_{.j.}\right\}_{i=1}^{i=a}\right\}_{j=1}^{j=b}$$

and so

$$\left\{\left\{\hat{\mu}_{ij}\right\}_{i=1}^{i=a}\right\}_{j=1}^{j=b} = \bigoplus_{j=1}^{b}\left[\bigoplus_{i=1}^{a} \frac{1}{n_{ij}} + (\beta/e)\underset{\sim}{J}_a\right]\left\{\left\{y_{ij.} - \beta n_{ij}\lambda_j y_{.j.}\right\}_{i=1}^{i=a}\right\}_{j=1}^{j=b} \ ;$$

i.e.,

$$\hat{\mu}_{ij} = y_{ij.}/n_{ij} + (\beta/e)y_{.j.} - \beta\lambda_j y_{.j.} - (\beta/e)\beta n_{.j}\lambda_j y_{.j.}$$

$$= \bar{y}_{ij.} + \beta y_{.j.}\left[\frac{1}{e} - \frac{1}{e + n_{.j}\beta} - \frac{n_{.j}\beta}{e(e + n_{.j}\beta)}\right]$$

$$= \bar{y}_{ij.} \; . \tag{55}$$

Hence in the 2-way cross-classification mixed model, with unbalanced data, the estimator of the cell mean μ_{ij} is the sample cell $\bar{y}_{ij}$ — a not unexpected result.

b. Including a within-cell covariance

Suppose that the variance-covariance structure of (11) also includes the within-cell covariance

$$\text{cov}(y_{ijk}y_{ijk'}) = \sigma_\gamma^2 \equiv \gamma \quad \text{for } k \neq k', \; \forall \; i \text{ and } j \; .$$

Denote the resulting var($\underset{\sim}{y}$) as $\underset{\sim}{V}_\gamma$. Then for $\underset{\sim}{V}$ of (14)

$$\underset{\sim}{V}_\gamma = \underset{\sim}{V} + \gamma \bigoplus_{i=1}^{a} \bigoplus_{j=1}^{b} \underset{\sim}{J}_{n_{ij}}$$

$$= \underset{\sim}{V} + \gamma\left(\bigoplus_{i=1}^{a} \bigoplus_{j=1}^{b} \underset{\sim}{1}_{n_{ij}}\right)\left(\bigoplus_{i=1}^{a} \bigoplus_{j=1}^{b} \underset{\sim}{1}'_{n_{ij}}\right)$$

$$= \underset{\sim}{V} + \gamma\underset{\sim}{X}\underset{\sim}{X}'$$

for $\underset{\sim}{X}$ of (48). Using (17) of Searle (1982), p. 261 again,

$$\underset{\sim}{V}_\gamma^{-1} = (\underset{\sim}{V} + \gamma\underset{\sim}{X}\underset{\sim}{X}')^{-1} = \underset{\sim}{V}^{-1} - \underset{\sim}{V}^{-1}\underset{\sim}{X}\underset{\sim}{M}^{-1}\underset{\sim}{X}'\underset{\sim}{V}^{-1} \tag{56}$$

for

$$\underset{\sim}{M} = (1/\gamma)\underset{\sim}{I} + \underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X} \; .$$

Denoting by $\hat{\underset{\sim}{\mu}}$ the solution in (55), we know that

$$\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X}\hat{\underset{\sim}{\mu}} = \underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{y} \ . \tag{57}$$

And letting the solution using $\underset{\sim}{V}_\gamma^{-1}$ be $\hat{\underset{\sim}{\mu}} + \hat{\underset{\sim}{\tau}}$ we need to solve

$$\underset{\sim}{X}'\underset{\sim}{V}_\gamma^{-1}\underset{\sim}{X}(\hat{\underset{\sim}{\mu}} + \hat{\underset{\sim}{\tau}}) = \underset{\sim}{X}'\underset{\sim}{V}_\gamma^{-1}\underset{\sim}{y}$$

for $\hat{\underset{\sim}{\tau}}$. Using (56), this equation is

$$(\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X} - \underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X}\underset{\sim}{M}^{-1}\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})(\hat{\underset{\sim}{\mu}} + \hat{\underset{\sim}{\tau}}) = \underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{y} - \underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X}\underset{\sim}{M}^{-1}\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{y}. \tag{58}$$

With (57), we find that (58) reduces to

$$-\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X}\underset{\sim}{M}^{-1}\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{y} + (\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X} - \underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X}\underset{\sim}{M}^{-1}\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})\hat{\underset{\sim}{\tau}} = -\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{M}^{-1}\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{y}$$

i.e.,

$$(\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X} - \underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X}\underset{\sim}{M}^{-1}\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X})\hat{\underset{\sim}{\tau}} = 0 \ . \tag{59}$$

Since the matrix in (59) is $\underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{X}$ and is presumed to be non-singular, the solution to (59) is $\hat{\underset{\sim}{\tau}} = \underset{\sim}{0}$. Hence $\hat{\underset{\sim}{\mu}}$ of (55), where there is no within-cell covariance σ_γ^2, is also the solution vector when there is a within-cell covariance. Thus in both cases $\hat{\mu}_{ij} = \bar{y}_{ij.}$ is the estimator of μ_{ij} — as might well be expected.

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APPENDIX: Analysis of BIB Data

a. Reconciliation of $\hat{\mu}_i$ with Scheffé.

One of the few places where the randomness of the blocks in a BIB design has been taken into account in estimating treatment effects is in Scheffé (1959) at pages 165-178. We show that the result given there, for estimation using recovery of interblock information, is consistent with $\hat{\mu}_i$ of (46). We begin with laying out equivalent notation.

Scheffé			This paper
p. 161:	# of treatments	I	$a = t$
	# of blocks	J	b
	# of replications	r	r
	block size	k	k
p. 162: (line 3 up)	# of occurrences of treatment i in block j	$K_{ij} = 0$ or 1	n_{ij}
p. 164: (lines 8-9)	i'th treatment total	g_i	$y_{i\cdot}$
	j'th block total	h_j	$y_{\cdot j}$
	i'th adjusted treatment total	$\mathscr{G}_i$	
(after 5.2.9):	$\mathscr{G}_i = g_i - k^{-1}\Sigma_j K_{ij} h_j$		$y_{i\cdot} - \Sigma_j n_{ij}\bar{y}_{\cdot j\cdot}$ $= y_{i\cdot} - r\bar{y}_{i(j)}$
	sum of block totals in which treatment i occurs	T_i	
(5.2.10):	$T_i = \Sigma_j n_{ij} h_j$		$kr\bar{y}_{i(j)}$
p. 166: (5.2.17)	efficiency factor	$\mathscr{E}$	
	$\mathscr{E} = \dfrac{rk - r + \lambda}{rk} = \dfrac{(k-1)I}{k(I-1)}$		$\dfrac{\lambda a}{rk} = \dfrac{(k-1)a}{k(a-1)}$

p. 165: (last line)	$r\delta\hat{\alpha}_i = G_i$	
	$\hat{\alpha}_i = G_i/r\delta$	$\dfrac{y_{i\cdot} - r\bar{y}_{i(j)}}{r\delta}$
		$= rk(\bar{y}_{i\cdot} - \bar{y}_{i(j)})/\lambda a$
p. 172: (5.2.33)	$\hat{\alpha}_i' = \dfrac{T_i - rJ^{-1}\Sigma_j h_j}{r - \lambda}$	$\dfrac{kr\bar{y}_{i(j)} - r\Sigma_j y_{\cdot j}/b}{r - \lambda}$
		$= \dfrac{kr\bar{y}_{i(j)} - ry_{\cdot\cdot}/b}{r - \lambda}$
		$= \dfrac{kr(\bar{y}_{i(j)} - \bar{y}_{\cdot\cdot})}{r - \lambda}$
(5.2.32b):	$\sigma_f^2 = k^2\sigma_B^2 + k\sigma_e^2$	$k(e + k\beta)$
(line 5 up):	$\psi = \Sigma_i c_i \alpha_i$ $\Sigma_i c_i = 0$	
	$\hat{\psi}' = \Sigma_i c_i \hat{\alpha}_i'$	
p. 174: (5.2.41)	$w = r\delta/\sigma_e^2$	$\lambda a/ke$
	$w' = (r - \lambda)/\sigma_f^2$	$(r - \lambda)/k(e + k\beta)$
p. 175: (5.2.42)	$\psi^* = \dfrac{w\hat{\psi} + w'\hat{\psi}'}{w + w'}$	

ψ^* is described by Scheffé as being unbiased and having minimum variance. It therefore corresponds to our $\hat{\underset{\sim}{\mu}}$. Since ψ is a contrast of α_i's it is also a contrast of $(\mu + \alpha_i)$ terms. The consistency of ψ^* with $\hat{\underset{\sim}{\mu}}$ will therefore be shown by adapting ψ^* to be

$$\tilde{\mu}_i = \frac{w(\hat{\mu} + \hat{\alpha}_i) + w'(\hat{\mu}' + \hat{\alpha}_i')}{w + w'}$$

and showing that $\tilde{\mu}_i = \hat{\mu}_i$.

Scheffé gives $\hat{\alpha}_i$ on page 165 — as shown above. Nowhere there does he show the corresponding $\hat{\mu}$. But in the last line of page 164 he mentions the "correction term for the grand mean". From that we infer that

$$\hat{\mu} = \bar{y}_{..} .$$

The expression for $\hat{\alpha}'_i$ is given at (5.2.34) on page 172. From (5.2.33) we get the corresponding

$$\hat{\mu}' = k\Sigma_j h_j / k^2 J = \Sigma_j y_{.j}/ka = \bar{y}_{..} .$$

Thus, using $\hat{\mu} = \hat{\mu}' = \bar{y}_{..}$ and w, w', $\hat{\alpha}$, $\hat{\alpha}'$ as above we have, from Scheffé's methodology,

$$\hat{\mu}_i = \bar{y}_{..} + \frac{\frac{\lambda a}{ke}\frac{kr}{\lambda a}\left(\bar{y}_{i.} - \bar{y}_{i(j)}\right) + \frac{r - \lambda}{k(e + k\beta)}\frac{kr(\bar{y}_{i(j)} - \bar{y}_{..})}{r - \lambda}}{\frac{\lambda a}{ke} + \frac{r - \lambda}{k(e + k\beta)}}$$

$$= \bar{y}_{..} + \frac{r\left[\left(\bar{y}_{i.} - \bar{y}_{i(j)}\right)/e + \left(\bar{y}_{i(j)} - \bar{y}_{..}\right)/(e + k\beta)\right]}{[\lambda a(e + k\beta) + (r - \lambda)e]/ke(e + k\beta)}$$

$$= \bar{y}_{..} + \frac{rk\left[(e + k\beta)\left(\bar{y}_{i.} - \bar{y}_{i(j)}\right) + e\left(\bar{y}_{i(j)} - \bar{y}_{..}\right)\right]}{\lambda ak\beta + rke},$$

because $\lambda a + r - \lambda = rk$

$$= \bar{y}_{..} + \frac{r(e + k\beta)}{re + a\lambda\beta}\left[\bar{y}_{i.} - \frac{k\beta}{e + k\beta}\bar{y}_{i(j)} - \frac{e}{e + k\beta}\bar{y}_{..}\right]$$

$$= \frac{r(e + k\beta)}{re + a\lambda\beta}\left[\bar{y}_{i.} - \frac{k\beta}{e + k\beta}\bar{y}_{i(j)} + \frac{a\lambda\beta}{r(e + k\beta)}\bar{y}_{..}\right],$$

which is (46).

b. The variance of $\hat{\mu}_i$

From (46)

$$v(\hat{\mu}_i) = v\left\{\frac{r(e + k\beta)}{re + \lambda a\beta}\left[\bar{y}_{i\cdot} - \frac{k\beta}{e + k\beta}\,\bar{y}_{i(j)} + \frac{\lambda a\beta}{r(e + k\beta)}\,\bar{y}_{\cdot\cdot}\right]\right\}$$

$$= \frac{r^2(e + k\beta)^2}{(re + \lambda a\beta)^2}\left\{v(\bar{y}_{i\cdot}) + \frac{k^2\beta^2}{(e + k\beta)^2}\,v(\bar{y}_{i(j)}) + \frac{\lambda^2 a^2\beta^2}{r^2(e + k\beta)^2}\,v(\bar{y}_{\cdot\cdot})\right.$$

$$+ 2\left[-\frac{k\beta}{e + k\beta}\,\mathrm{cov}(\bar{y}_{i\cdot},\bar{y}_{i(j)}) - \frac{k\beta}{e + k\beta}\,\frac{\lambda a\beta}{r(e + k\beta)}\,\mathrm{cov}(\bar{y}_{i(j)},\bar{y}_{\cdot\cdot})\right.$$

$$\left.\left. + \frac{\lambda a\beta}{r(e + k\beta)}\,\mathrm{cov}(\bar{y}_{i\cdot},\bar{y}_{\cdot\cdot})\right]\right\}$$

$$= \frac{r^2(e + k\beta)^2}{(re + \lambda a\beta)^2}\left\{\frac{r(e + \beta)}{r^2} + \frac{k^2\beta^2 rk(e + k\beta)}{(e + k\beta)^2 r^2 k^2} + \frac{\lambda^2 a^2\beta^2 ar(e + k\beta)}{r^2(e + k\beta)^2 a^2 r^2}\right.$$

$$\left. + 2\left[\frac{-k\beta}{e + k\beta}\,\frac{r(e + k\beta)}{rrk} - \frac{\lambda ka\beta^2}{r(e + k\beta)^2}\,\frac{kr(e + k\beta)}{krar} + \frac{\lambda a\beta r(e + k\beta)}{r(e + k\beta)rar}\right]\right\}$$

$$= \frac{e + k\beta}{(re + \lambda a\beta)^2}\left\{r(e+\beta)(e+k\beta) + rk\beta^2 + \lambda^2 a\beta^2/r + 2[(-r\beta+\lambda\beta)(e+k\beta) - k\lambda\beta^2]\right\}$$

$$= \frac{(e + k\beta)}{(re + \lambda a\beta)^2}\left\{re^2 + \beta^2(rk+rk+\lambda^2 a/r-2rk+2\lambda k-2k\lambda) + \beta e(r+rk-2r+2\lambda)\right\}$$

$$= \frac{(e + k\beta)}{(re + \lambda a\beta)^2}\left[re^2 + \frac{\lambda^2 a}{r}\,\beta^2 + \beta e(rk - r + 2\lambda)\right]$$

$$= \frac{(e + k\beta)}{(re + \lambda a\beta)^2}\,[r^2e^2 + r\lambda(a + 1)\beta e + \lambda^2 a\beta^2]/r, \text{ because } rk-r+2\lambda = \lambda(a + 1)$$

$$= \frac{(e + k\beta)}{r(re + \lambda a\beta)^2}\,(re + \lambda a\beta)(re + \lambda\beta)$$

$$= \frac{(e + k\beta)(re + \lambda\beta)}{r(re + \lambda a\beta)}, \text{ which is (47).}$$